The Application of Signal Detection Theory to Optics

PROGRESS REPORT

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ABSTRACT

Research undertaken during the quarter ending March 15, 1969, is outlined. Topics included are the quantum-limited detectability of an object emitting light with a Lorentz spectrum, the decision among three pure states in quantum detection theory, and the distribution of photoelectric counts from incoherent light.

1. Detection of Incoherent Objects by a Quantum-Limited Optical System

A paper[1] submitted with our previous progress report treated the detection of incoherent objects by observation of the electromagnetic field at the aperture of an optical instrument. This field results from thermal background radiation and, when the object is present, from light emitted by it. The problem is to design the optical instrument to detect the object with maximum reliability. The field is observed during an interval $(0, T)$ that is much longer than $1/W$, where W is the bandwidth of the light from the object. The average number of photons received from the background is taken to be so small that detection can be said to be quantum limited.

Under these conditions the best detector is an instrument that measures the quantum threshold operator, and the paper referred to showed that this operator is a quadratic functional of the field at the aperture A. It takes the form

$$U = \iint_{A\,A} d^2 \underset{\sim}{r}_1 \, d^2 \underset{\sim}{r}_2 \int_0^T \int_0^T dt_1 \, dt_2 \times$$

$$\Psi_- (\underset{\sim}{r}_1, t_1) \, \varphi_s (\underset{\sim}{r}_1, t_1; \underset{\sim}{r}_2, t_2) \, \Psi_+ (\underset{\sim}{r}_2, t_2),$$

where $\Psi_+(\underset{\sim}{r}, t)$ is the positive-frequency part of the field, assumed for simplicity to be a scalar, $\Psi_- (\underset{\sim}{r}, t) = [\Psi_+ (\underset{\sim}{r}, t)]^+$ is the negative-frequency part, and $\varphi_s (\underset{\sim}{r}_1, t_1; \underset{\sim}{r}_2, t_2)$ is the mutual coherence function of the field component due to the object. The value of U is compared with a decision level U_0, and when $U > U_0$ the object is declared to be present.

A procedure was given for calculating the false-alarm and detection probabilities for such a detector. These require knowing the probability density functions (p. d. f. 's) of U when background radiation alone is present and when both object and background are present. Only the moment-generating functions (m. g. f. 's) of U are simple to calculate, and finding its p. d. f. 's requires numerical computation in general. It was shown that when the spectrum of the object light is constant over a frequency band of width W and is zero outside it (a rectangular spectrum), the decision statistic U has a compound Poisson distribution.

A more realistic object spectrum is the Lorentz spectrum,

$$X(\omega) = 2w/(\omega^2 + w^2) ,$$

where w is the bandwidth and ω is the angular frequency referred to the central frequency of the object spectrum. For this spectrum the m. g. f. 's take a complicated form involving modified Bessel functions. Analytical inversion of the m. g. f. 's to obtain the p. d. f. 's of U appears impossible.

A numerical calculation of the false-alarm and detection probabilities for a Lorentz spectrum was carried out and is described in a paper attached to this report.[2] It was found that when the bandwidths are set equal, $w = W$, an object with a Lorentz spectrum has a lower probability of detection than one with a rectangular spectrum, for equal false-alarm probabilities and equal average numbers of photons received from object and background. It was assumed that the light possessed first order spatial coherence over the entire aperture.

2. Multi-Hypothesis Quantum Decision Theory

In ordinary statistical theory the optimum strategy for choosing among M hypotheses ($M \geq 2$) can be prescribed in terms of their posterior probabilities and their posterior risks, given the observed data. A solution of the corresponding problem in quantum decision theory has not been found for $M > 2$. Even the apparently simpler problem of how best to decide among M possible pure states of a quantum system has not been solved for $M > 2$.

In a search for a clue to the solution, the choice among three coherent states of a simple harmonic oscillator was investigated. The oscillator might represent a mode of an ideal quantum receiver unperturbed by thermal radiation. A computer program was written to determine the optimum projection operators for making the decision with minimum average probability of error, the three states being assigned equal prior probabilities. No obvious symmetries that might lead to a general solution were discovered. The method and detailed results are presented in the appendix of this report.

3. Other Tasks

A review of quantum detection and estimation theory has been written for the "Journal of Statistical Physics." A copy is attached to this report.[3]

An early paper[4] analyzed the distribution of photoelectric counts when partially polarized incoherent light falls on a photosensitive surface. The moment generating function of the number of counts during an interval of duration T was expressed in terms of the Fredholm determinant of an integral equation whose kernel is the temporal autocovariance function of the light. This Fredholm determinant has been calculated for light with a Lorentz spectrum. Under investigation now is the determination of the Fredholm determinant for light with a spectrum of the form

$$X(\omega) = \frac{A\omega^2 + B}{C\omega^4 + D\omega^2 + E} ,$$

with the aim of assessing the influence of the form of the spectrum on the distribution. State-space methods are being applied in this effort.

REFERENCES

1. C. W. Helstrom, "Detection of Incoherent Objects by a Quantum Limited Optical System," submitted to J. Opt. Soc. Am.

2. C. W. Helstrom, "Quantum-Limited Detection of an Incoherent Object with a Lorentz Spectrum," submitted to J. Opt. Soc. Am.

3. C. W. Helstrom, "Quantum Detection and Estimation Theory," submitted to J. Stat. Phys.

4. C. W. Helstrom, "The Distribution of Photoelectric Counts from Partially Polarized Gaussian Light," Proc. Phys. Soc. (London) 83, 777-782 (1964).

APPENDIX

Decision Among Pure States in Quantum Detection Theory

An outstanding problem in quantum detection theory is the optimum manner of choosing among M density operators $\rho_1, \rho_2, \ldots, \rho_M$. The operator ρ_k $(k = 1, 2, \ldots, M)$ describes the state of a quantum receiver when the k-th of M signals has been transmitted to it. The choice is to be made by measuring M commuting projection operators Π that form a resolution of the identity,

$$\Pi_1 + \Pi_2 + \ldots + \Pi_M = \underset{\sim}{1} . \tag{1.1}$$

Those M commuting operators Π_k will be adopted that minimize an average cost of operation.[3] In a communication system in which all errors are equally serious, and all signals are transmitted equally often, it is appropriate to choose the Π_k's to minimize the average probability P_e of error, or to maximize the average probability $\bar{Q}$ of correct decision,

$$\bar{Q} = 1 - P_e = \frac{1}{M} \sum_{k=1}^{M} \mathrm{Tr}\,(\rho_k \Pi_k). \tag{1.2}$$

It is not at present known how to determine the operators Π_k that maximize $\bar{Q}$, except when $M = 2$.

An apparently simpler problem, but one that is also unsolved, is to find the optimum operators Π_k for deciding among M pure states

$|\Psi_1\rangle, \ldots |\Psi_j\rangle, \ldots, |\Psi_M\rangle$. Then the density operators are

$$\rho_k = |\Psi_k\rangle\langle\Psi_k|, \quad k = 1, 2, \ldots, M, \tag{1.3}$$

and the projection operators will project on to M linear and orthonormal combinations $|\eta_j\rangle$ of the states $|\Psi_k\rangle$,

$$\Pi_j = |\eta_j\rangle\langle\eta_j|, \qquad \langle\eta_j|\eta_k\rangle = \delta_{jk}, \tag{1.4}$$

where

$$|\eta_j\rangle = \sum_{k=1}^{M} c_{jk}|\Psi_k\rangle . \tag{1.5}$$

The average probability of correct decision is now

$$\overline{Q} = 1 - P_c = \frac{1}{M}\sum_{k=1}^{M} |\langle\eta_k|\Psi_k\rangle|^2 . \tag{1.6}$$

When the M projection operators of Eq. (1.4) are measured, the system will be found in one of the M states $|\eta_1\rangle, \ldots |\eta_M\rangle$. If it is found in the state $|\eta_k\rangle$, the decision is made that the system was originally in state $|\Psi_k\rangle$. The probability when the original state was $|\Psi_k\rangle$ that this will happen is just $|\langle\eta_k|\Psi_k\rangle|^2$, and the average probability of correct decision, $\overline{Q}$ of Eq. (1.6), is the average of these probabilities over the equally likely states $|\Psi_k\rangle$. The problem is to choose the M orthogonal states $|\eta_k\rangle$ so that $\overline{Q}$ is maximum. It has been solved only for M = 2.

If all the scalar products $\langle \Psi_j | \Psi_k \rangle$ are real (j, k = 1, 2, . . . , M), the states $|\Psi_k\rangle$ can be represented as unit vectors in an M-dimensional Euclidean space. They are not in general orthogonal. (If they are, our problem is easy.) The $|\eta_k\rangle$ similarly form an orthonormal set of vectors in the same space. Let γ_k be the angle between $|\eta_k\rangle$ and $|\Psi_k\rangle$. Then

$$\bar{Q} = M^{-1} \sum_{k=1}^{M} \cos^2 \gamma_k \quad . \tag{1.7}$$

The problem is to place the orthonormal set $|\eta_k\rangle$ in such a position that it lies as close as possible to the set of vectors $|\Psi_k\rangle$, the "distance" between the sets being measured in a reverse sense by $\bar{Q}$; the smaller the angles γ_k , the larger $\bar{Q}$.

In an attempt to see whether the optimum position of the orthonormal vectors $|\eta_k\rangle$ with respect to the $|\Psi_k\rangle$'s exhibits any symmetries that might help to solve the problem in general, the three-dimensional problem was studied by computer. For M = 3 the ends of the vectors $|\Psi_1\rangle$, $|\Psi_2\rangle$, and $|\Psi_3\rangle$ form a spherical triangle $A_1 A_2 A_3$ on the unit sphere, and the ends of the orthonormal vectors $|\eta_k\rangle$ form a polar triangle $P_1 P_2 P_3$ whose sides and angles equal 90^0. (See Fig. 1.) The problem is then one of orienting the triangle $A_1 A_2 A_3$ in such a way that its vertices lie as close as possible to P_1, P_2, and P_3, respectively, in the sense that

$$\bar{Q} = \frac{1}{3}\ (\cos^2 \gamma_1 + \cos^2 \gamma_2 + \cos^2 \gamma_3) \tag{1.8}$$

must be maximum, or

$$P_e = \frac{1}{3}\ (\sin^2 \gamma_1 + \sin^2 \gamma_2 + \sin^2 \gamma_3) \tag{1.9}$$

must be minimum.

As the states $|\Psi_k\rangle$ we pick three coherent states $|\alpha_1\rangle$, $|\alpha_2\rangle$, and $|\alpha_3\rangle$ of a simple harmonic oscillator, which might represent a single mode of the electromagnetic field of the cavity of an ideal receiver.[3] The scalar products that specify the angles φ_{ij} between the vectors are then

$$\cos \varphi_{ij} = \langle \alpha_i \mid \alpha_j \rangle = \exp\ (\alpha_i^* \alpha_j - \frac{1}{2}\ |\alpha_i|^2 - |\alpha_j|^2\ ), \tag{1.10}$$

and in order for these to be real the complex numbers α_i must be real,

$$\cos \varphi_{ij} = \exp \left[-\frac{1}{2}\ (\alpha_i - \alpha_j)^2 \right] . \tag{1.11}$$

These numbers are so defined that

$$N_i = |\alpha_i|^2 \tag{1.12}$$

is the average number of signal photons in the mode when it is in the state $|\alpha_i\rangle$.

Thus our problem corresponds to detecting one of three signals of different amplitudes. The field mode has an effective temperature of absolute zero; it contains no thermal noise. Error in the decision arises only from the quantum-mechanical uncertainty of the coherent states $|\alpha_i\rangle$, $i = 1, 2, 3$. The angles φ_{12}, φ_{23}, and φ_{31} define arcs on the unit sphere that form the sides a_3, a_1, and a_2 of the spherical triangle whose optimum position with respect to the polar triangle $P_1 P_2 P_3$ we wish to find.

2. The Spherical-Trigonometrical Problem

Given the spherical triangle $A_1 A_2 A_3$ with sides a_1, a_2, a_3, the problem is to orient it with respect to the polar triangle $P_1 P_2 P_3$ so that

$$\bar{Q} = \frac{1}{3} (\cos^2 A_1 P_1 + \cos^2 A_2 P_2 + \cos^2 A_3 P_3) \tag{2.1}$$

is maximum. (See Fig. 1).

The location of the triangle $A_1 A_2 A_3$ with respect to P_1, P_2, and P_3 is specified by the arcs $OP_1 = s_1$, $OP_2 = s_2$, $OP_3 = s_3$ from the center O of the circumscribed circle of $\Delta A_1 A_2 A_3$, with

$$\cos^2 s_1 + \cos^2 s_2 + \cos^2 s_3 = 1 . \tag{2.2}$$

The orientation of the triangle is specified by the angles $\angle P_1 O A_1 = \psi_1$, $\angle P_2 O A_2 = \psi_2$, $\angle P_3 O A_3 = \psi_3$ between the arcs OP_1, OP_2, OP_3 and the radii OA_1, OA_2, OA_3 from the circumcenter O to the vertices of the triangle. The arc lengths of these radii are equal to R, which is given by

$$\tan R = \left[- \frac{\cos S}{\cos (S-A_1) \cos (S-A_2) \cos (S-A_3)} \right]^{1/2},$$

$$S = \frac{1}{2} (A_1 + A_2 + A_3), \qquad (2.3)$$

where A_1, A_2, A_3 are the angles of the triangle $A_1 A_2 A_3$. The radii make angles θ_1, θ_2, θ_3 with each other, given by

$$\sin (\tfrac{1}{2} \theta_i) = \sin (\tfrac{1}{2} a_i) \; / \sin R, \quad i = 1, 2, 3. \qquad (2.4)$$

The angles between the radii OP_1, OP_2, OP_3 are $\angle P_1OP_2 = \varphi_3$, $\angle P_2 OP_3 = \varphi_1$, $\angle P_3 OP_2 = \varphi_2$, given by

$$\begin{aligned} \cos \varphi_1 &= - \cot s_2 \cot s_3 \\ \cos \varphi_2 &= - \cot s_3 \cot s_1 \\ \cos \varphi_3 &= - \cot s_1 \cot s_2 \; . \end{aligned} \qquad (2.5)$$

It suffices to give s_1, s_2 and ψ_1 in order to fix the location of the triangle $A_1 A_2 A_3$. In terms of these,

$$\begin{aligned} \psi_2 &= \psi_1 + \theta_3 - \varphi_3 , \\ \psi_3 &= \psi_1 - \theta_2 + \varphi_2 . \end{aligned} \qquad (2.6)$$

The connecting arcs are given by

$$\cos A_iP_i = \cos s_i \cos R + \sin s_i \sin R \cos \psi_i, \; i = 1, 2, 3, \qquad (2.7)$$

and the quantity to be maximized is

$$\bar{Q}(s_1, s_2, \psi_i) = \frac{1}{3} \sum_{i=1}^{3} \cos^2 A_i P_i =$$

$$\frac{1}{3} (\cos^2 R + 2M \sin R \cos R + N \sin^2 R),$$

$$M = \sum_{i=1}^{3} \sin s_i \cos s_i \cos \psi_i . \qquad (2.8)$$

$$N = \sum_{i=1}^{3} \sin^2 s_i \cos^2 \psi_i .$$

For any given position (s_1, s_2) of the circumcenter O, the function $\bar{Q}(s_1, s_2, \psi_1)$ was maximized with respect to ψ_1 by solving

$$\frac{\partial \bar{Q}}{\partial \psi_1} = 0 \qquad (2.9)$$

by Newton's method, producing a maximum value $\bar{Q}'(s_1, s_2) = \max_{\psi_1} \bar{Q}(s_1, s_2, \psi_1)$.

The value of s_1 was then changed to $s_1 + \delta$ and $s_1 - \delta$ and the maxima $\bar{Q}'(s_1 + \delta, s_2)$ and $\bar{Q}'(s_1 - \delta, s_2)$ were calculated in the same way. A quadratic function was fitted to the three values $q_1 = \bar{Q}'(s_1 + \delta, s_2)$, $q_0 = \bar{Q}'(s_1, s_2)$, and $q_{-1} = \bar{Q}'(s_1 - \delta, s_2)$; and the maximum of this quadratic function was located at

$$s_1' = s_1 - \left(\frac{q_1 - q_{-1}}{q_1 + q_{-1} - 2q_0} \right) \frac{\delta}{2}, \qquad (2.10)$$

which became the s_1-coordinate of the new trial point. The same procedure was applied to s_2, and a new value of s_2 obtained from the values of $\bar{Q}'(s_1, s_2 + \delta)$, $\bar{Q}'(s_1, s_2)$, and $\bar{Q}'(s_1, s_2 - \delta)$. The value of δ was then reduced by a factor of 1/5 and the procedure repeated, first for s_1 and second for s_2. In this way the triangle $A_1 A_2 A_3$ was moved about on the sphere until the values of $\bar{Q}$ ceased to change by more than 10^{-5}.

The computer printed out the final location and orientation of the triangle $A_1 A_2 A_3$ and the lengths of the arcs $A_1 P_1$, $A_2 P_2$, $A_3 P_3$. Unfortunately, no symmetries were apparent that might indicate a general principle governing the optimum configuration.

3. Comparison with the Quasi-Classical Detector

The quasi-classical or threshold detector for choosing between the three coherent states described by eqs. (1.10-.12) measures the operator $\frac{1}{2}(a + a^+)$, where a and a^+ are the annihilation and creation operators for the mode.[3] The outcome q of this measurement is a Gaussian random variable with mean values

$$\langle \alpha_i | \frac{1}{2} (a + a^+) | \alpha_i \rangle = R\ell \ \alpha_i, \ i = 1, 2, 3, \tag{3.1}$$

in the three states, and with variances 1/4. With α_1, α_2, α_3 real and so ordered that $\alpha_1 < \alpha_2 < \alpha_3$, the quasi-classical detector chooses $|\alpha_1\rangle$ as the true state when $q < \frac{1}{2}(\alpha_1 + \alpha_2)$, $|\alpha_2\rangle$ when $\frac{1}{2}(\alpha_1 + \alpha_2) < q < \frac{1}{2}(\alpha_2 + \alpha_3)$, and $|\alpha_3\rangle$ when $q > \frac{1}{2}(\alpha_2 + \alpha_3)$. The

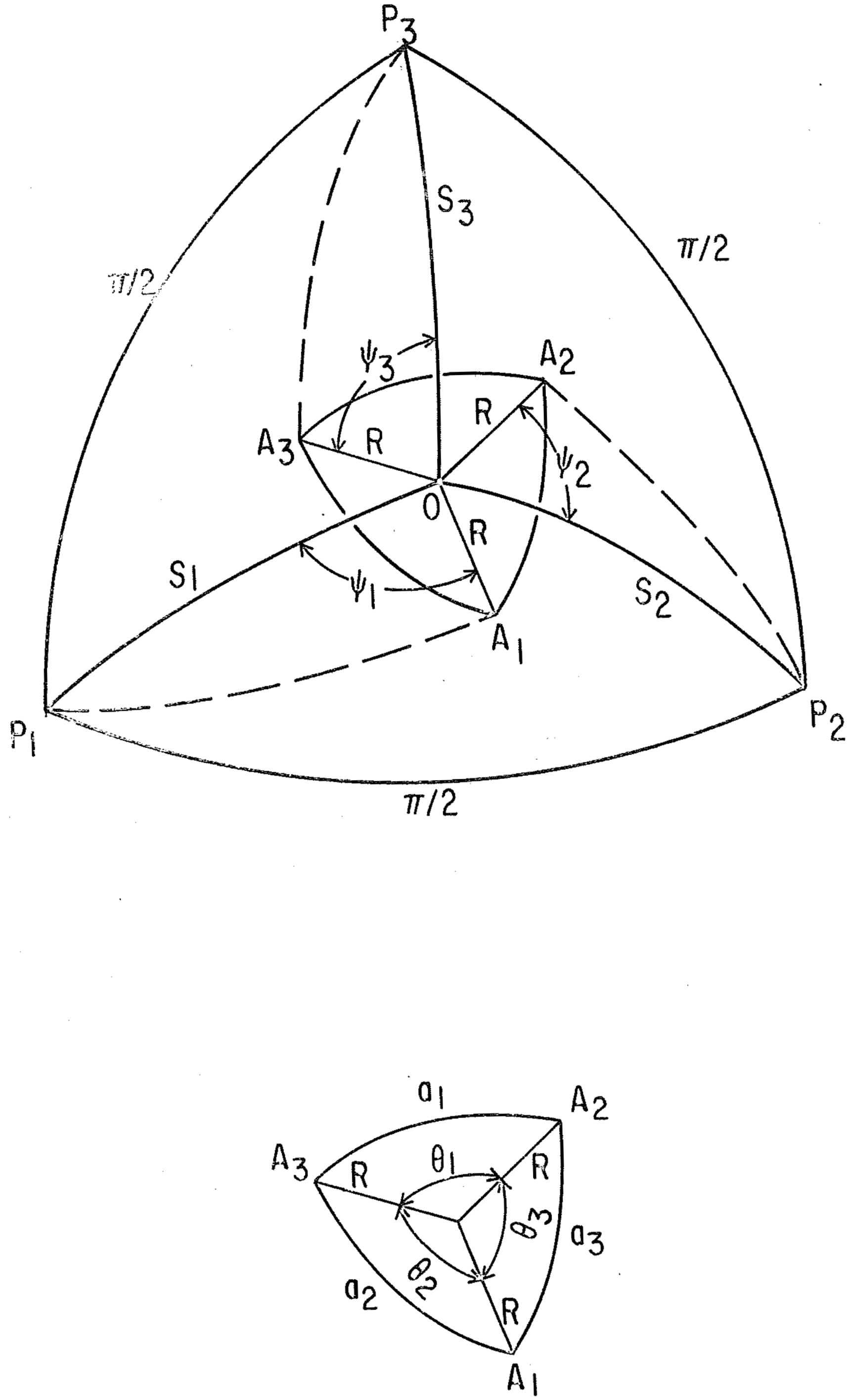

FIG. 1. The Spherical Triangles

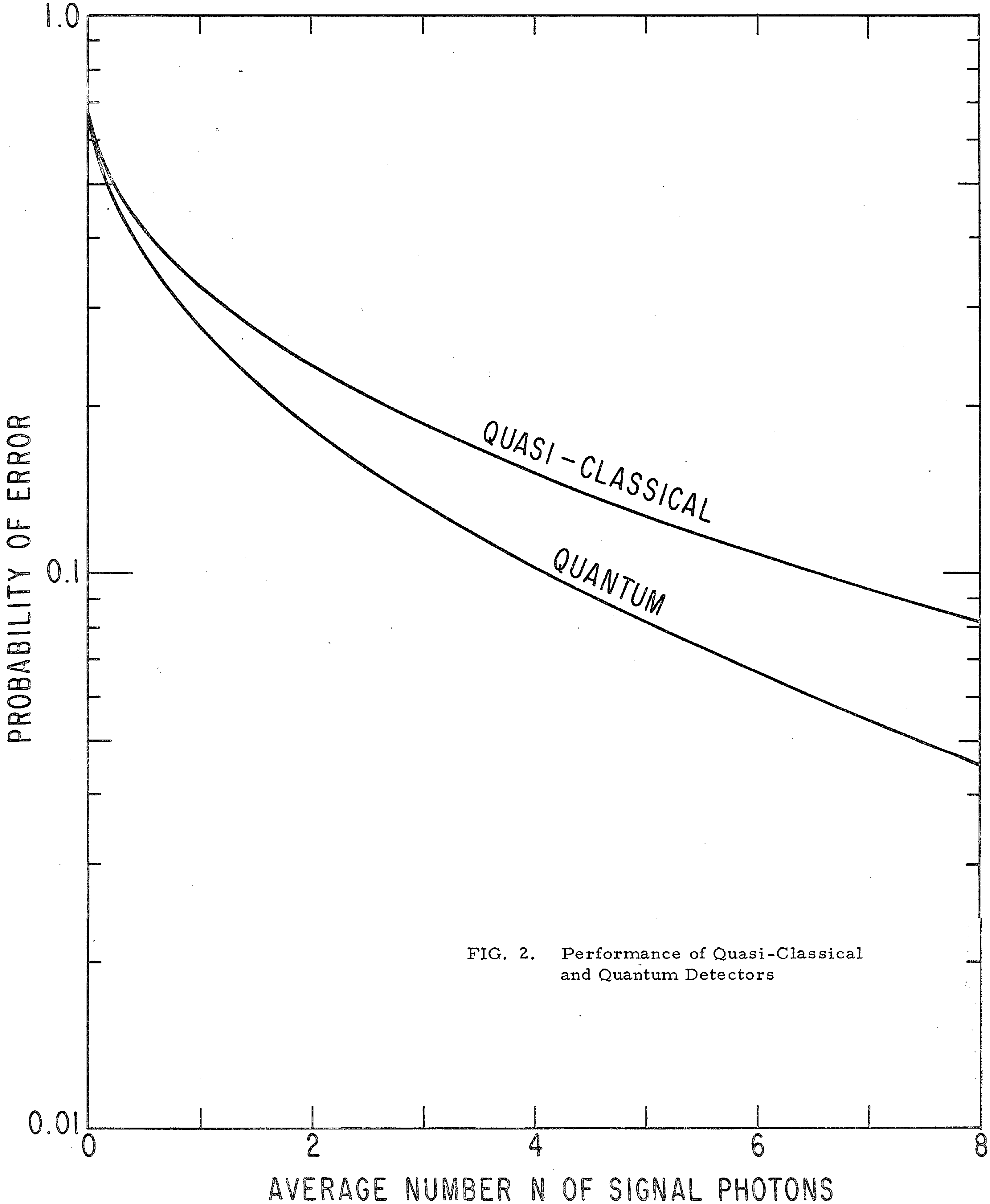

FIG. 2. Performance of Quasi-Classical and Quantum Detectors